

Lattices of Knowledge Systems

Peeter Lorents, Enn Tyugu

Cooperative Cyber Defense Center of Excellence
 Filtri tee 12 Tallinn 10132, Estonia

Abstract - An abstract form of knowledge representation is used and knowledge lattices are introduced. A formal concept of knowledge system has been defined independently of its implementation in two different but equivalent ways: first, as a free deductive system, and second, as an algebraic system with notation-denotation relation supplied with derivation rules. Operations on knowledge systems are analyzed. Lattices of knowledge systems and abstract goals on knowledge systems are described and used for analysis of solvability of goals.

Keywords: Knowledge representation, deductive system, knowledge system, semantic system.

1 Introduction

As the knowledge-based systems are growing in size and complexity, it has become necessary to describe them on a more abstract level than their implementation. It is reasonable to use algebraic structures and logical means for definition and analysis of knowledge and knowledge-based systems. The present work considers modularity of knowledge-based systems, and focuses on properties of their building blocks – knowledge modules. We call such a module a knowledge system, and represent it as a deductive system. There is an evidence that knowledge used by experts is modular [2]. Also complex knowledge-based software has modular knowledge architecture [7,3]. The aim of this work is to analyze a structure of a set of possible knowledge systems on an abstract level. The main result of this analysis is that knowledge systems constitute lattices with interesting properties that are related to the knowledge handling capabilities – to the solvability of goals.

We begin the analysis with definitions of knowledge and semantic systems. The latter can be considered as pre-ontologies, i.e. the ontologies without the usage of relations between the concepts. An example of a simple definition of knowledge is the one given by M. Firebaugh [1]: “Knowledge is information in a meaningful context”. A. Newell, R. Young and T. Poek give another definition of knowledge [8] that is more detailed, and applicable only in a specific context: “Knowledge is a system of patterns within the bulk the available units of information based upon experiences; these

patterns are generalized, labeled and organized, into a relational network with a particular architecture, knowledge implies one ability to use it efficiently for reasoning and decision making.” We give a formal definition of knowledge in Section 2, where the notions of notation-denotation relation and semantic systems are introduced and some simple theorems about the semantic systems are presented.

Sections 3, 4 and 5 are dedicated to knowledge systems, relations between them and operations on knowledge systems. Our definition of a knowledge system relies on the ideas of S. Maslov presented in his book [6], where he demonstrates the usage of deductive systems for knowledge handling in several domains. Section 6 introduces goals as the means of accessing knowledge in a knowledge system and describes a lattice of goals. Section 7 includes the main results – a definition of lattices of knowledge systems and four theorems about these lattices.

2 Knowledge representation and semantic systems

In the present work we rely on a fundamental binary relation – *notation-denotation relation* between the sets of notations and denotations [5]. We are going to use the symbol \int as a notation of the notation-denotation relation. If a meaning (i.e. the denotation) of A is B , then we write $A\int B$.

Definition 1. An ordered pair $\langle A, B \rangle$ is called knowledge, iff $A\int B$ [4].

Remark. The pair in the definition above can be considered as a way of knowledge representation. In [5] Lorents analyzes its relatedness to several other knowledge representation forms.

We consider often a set of notations S and a set of denotations (i.e. meanings) M , when speaking about knowledge. This gives us an idea to use an algebraic system for analyzing knowledge representation.

Definition 2. An algebraic system $\langle S, M; \int \rangle$ where S and M are non-empty sets of notations and denotations, and \int is a notation-denotation relation between S and M is a *semantic system*.

Remark. A semantic system can be called also a pre-ontology – that is an ontology without derivation possibilities. To get an ontology from a pre-ontology, one has to add derivation rules that enable one to reason about the knowledge elements in a suitable way. For instance, at least rules of inheritance should be added in order to get an ontology in the conventional sense.

It may happen that we have to extend a set of denotations M of a semantic system by introducing new elements, say the elements of a set Q . Then we need at least one new notation that we can call, for instance, *something* to denote the new meanings. In this way we get a new semantic system.

Definition 3. The semantic system $\langle S', M'; f' \rangle$ is called a primitive extension of a semantic system $\langle S, M; f \rangle$ if $S' = S \cup \{\text{something}\}$, $M' - M \neq \emptyset$, $M \subseteq M'$ and $f' = f \cup \{(\text{something}, y) \mid y \in M' - M\}$. (The notation-denotation relation has been extended with pairs $\langle \text{something}, y \rangle$ for all new meanings y .)

It is sometimes needed to create a new semantic system from existing ones by joining the existing systems. This can be done by an operation called *sum*.

Definition 4. A sum $\langle S', M'; f' \rangle$ of semantic systems $\langle S_1, M_1; f_1 \rangle$ and $\langle S_2, M_2; f_2 \rangle$ is the semantic system with $S' = S_1 \cup S_2$, $M' = M_1 \cup M_2$ and $f' = f_1 \cup f_2$. The sum operation is denoted by \oplus .

Theorem 1. The sum of semantic systems is an idempotent, commutative and associative operation.

Proof. Follows from the same properties of the union operation of sets, taking into account the definition of sum.

Corollary. Any set of semantic systems that is closed for the operation of sum is a semilattice.

Remark. Sum of semantic systems may have an unwanted property that some notation will denote contradictory or discrepant meanings. This happens, if the notation is used in the two initial semantic systems in „different meaning”, i.e. it is used for different purposes. This can be said more precisely as follows.

Example. One cook can denote for brevity salt by s in a recipe. But another cook can use the same s for denoting sugar. Taking a sum of their semantic systems would be disastrous.

Definition 5. A semantic system is *consistent*, if its notation-denotation relation does not give contradictory meanings for any notation. Semantic systems are *harmonious*, if their notation-denotation relations do not give contradictory

meanings on the union of their sets of meanings for any notation.

Remark. This definition can be made precise by introducing a noncontradiction relation C on the union of the sets of meanings. This relation is reflexive and symmetric. If it is transitive as well, it is an equivalence relation, Being consistent requires that a notation always denotes meanings bound by this relation. Being consistent is expressed for a semantic system by the formula

$$\forall s((\exists m_1, \exists m_2, s f m_1 \& s f m_2) \supset C(m_1, m_2)),$$

and being harmonious is expressed for two semantic systems $\langle S_1, M_1; f_1 \rangle$ and $\langle S_2, M_2; f_2 \rangle$ by the formula

$$\forall s((\exists m_1, \exists m_2, s f_1 m_1 \& s f_2 m_2) \supset C(m_1, m_2)).$$

Definition 6. Harmonization of consistent semantic systems $\langle S_1, M_1; f_1 \rangle$ and $\langle S_2, M_2; f_2 \rangle$ is building new harmonious semantic systems $\langle S', M'; f' \rangle$ and $\langle S'', M''; f'' \rangle$, where $M_1 = M'$, $M_2 = M''$, and any disharmonious notation in $S_1 \cap S_2$ will be changed to two new different notations.

Corollary. Harmonization gives new harmonious semantic systems with the unchanged sets of denotations.

Remark. The method of harmonization described by Definition 6 can be applied to a single semantic system as well, in order to make it consistent.

Theorem 2. If the sets of notations of consistent semantic systems do not have common elements, then the semantic systems are harmonious, and so is their sum.

Proof. Consistent semantic systems can be disharmonious only if they have common notations.

Definition 7. A product $\langle S', M'; f' \rangle$ of semantic systems $\langle S_1, M_1; f_1 \rangle$ and $\langle S_2, M_2; f_2 \rangle$ is the semantic system with $S' = S_1 \cap S_2$, $M' = M_1 \cap M_2$ and $f' = f_1 \cap f_2$. The product operation is denoted by the symbol \otimes .

Theorem 3. The product of semantic systems is an idempotent, commutative and associative operation.

Proof. Follows from the same properties of the intersection of sets.

Theorem 4. A set of semantic systems closed for operations of sum and products is a lattice with a zero element.

Proof. Follows from Theorems 1 and 3 and the definition of a lattice.

3 Knowledge systems

Semantic systems describe knowledge representation in an abstract way. Now we are going to add abstract description of knowledge handling. The aim of this section is to define the concept of a knowledge system that includes both knowledge representation and knowledge handling features. We are going to do this in a way that is independent of any particular implementation, and is usable also in the analysis and explanation of architectures of knowledge-based applications. Unlike semantic systems, a knowledge system must include means for knowledge handling – for deriving new notations from given ones. This leads us to a concept of the deductive system. For detailed discussion of this formalism we refer to Maslov [6].

Conventional concept of a *deductive system* is defined as a language for representing objects, a set of initial objects and derivation rules for generating new objects from given objects. (See, for instance, Maslov [6].) We are going to use a slightly different concept – a free deductive system, where initial objects are not fixed. Giving some initial objects in the language of a free deductive system makes it a conventional deductive system.

As we have shown already, notations for knowledge, called here also knowledge objects have some meaning that may be objects or maybe even some effects on the environment where they exist. To make it precise, one has to consider a collection of possible meanings and a mapping from notation to meanings. This can be formalized as an *interpretation* of notations. One can, instead of the mapping, consider a relation that binds notations with their meanings. This is a *relation of notation-denotation* defined in Section 2.

Definition 8. Free deductive system is a language of objects and rules for generating new objects from given objects.

Definition 9. Interpretation of a (free) deductive system is a set M of entities that are possible meanings of objects of the deductive system, and a relation f that binds at least one meaning with every object, and binds at least one object (a notation) with every meaning included in M . The relation f is called a *relation of notation-denotation* [7].

Definition 10. Knowledge system is an interpreted free deductive system [7].

We have already defined semantic systems that include a set of notations, a set of meanings (denotations) and a notation-denotation relation, therefore we can use this concept in a definition of knowledge system, and give the alternative definition of a knowledge system.

Definition 11. Knowledge system is a pair of a semantic system and a set of derivation rules that operate on the notations of the semantic system: $\langle S, M; f, D \rangle$, where S is a set of notations of the semantic system, M is a set of meanings

(denotations of the semantic system), f is a notation-denotation relation that defines interpretation of notations of the knowledge system, and D is a set of derivation rules on the set S .

The two definitions of a knowledge system are in all respects the same. The first uses explicitly the concept of a deductive system and follows the way laid out by S. Maslov, the second uses the concept of a semantic system and relates the knowledge systems to the former works of P. Lorents.

Examples of free deductive systems are natural languages with one derivation rule that generates a new sequence of sentences ST for any correct sequences of sentences S and T of a language. This rule can be written as follows:

$$\frac{S \quad T}{ST}$$

If we are able to get a meaning for any sequence of sentences, and can find a sentence denoting a given meaning, we can say that we have a knowledge system. In other words – we have a knowledge system, if and only if we understand the language.

Remark. A natural method to get a meaning of a new notation generated by a rule is to construct it from meanings of other notations used in the rule. In the present example of natural language texts, one should take the union of meanings of given texts and add the meaning of interdependencies of the texts, e.g. ellipsis in the second text etc.

4 Semantic connectedness and conservative extension of knowledge systems

Let us have knowledge systems K_1 and K_2 with sets of notations S_1, S_2 , sets of meanings M_1, M_2 , sets of derivation rules D_1, D_2 and notation-denotation relation f_1, f_2 respectively.

Several knowledge systems may have the sets of meanings with nonempty intersection. This is the case, for instance, with systems of classical logic that have different sets of inference rules, or programming languages for one and the same computer, and even natural languages for identical cultures. (The cultures depend to some extent on the language, hence there are no absolutely identical cultures with different languages, but the sets of meanings of the cultures intersect to a large extent.)

Definition 12. We say that knowledge systems are *semantically connected*, if they have the sets of meanings with nonempty intersection.

A degree of semantic connectedness of knowledge systems with measurable sets of meanings can be easily defined. A *degree of connectedness* of knowledge systems with the sets of meanings M_1 and M_2 and a measure μ given on the sets is

$$C(K_1, K_2) = \mu(M_1 \cap M_2) / \mu(M_1 \cup M_2),$$

where $\mu(H)$ denotes the measure of the set H . (It is natural to take the number of elements of a set as its measure in the case of finite sets.)

Definition 13. *Conservative extension* of a knowledge system K with set of notations S and set of meanings M on a set of meanings M' , $M \subset M'$, is a knowledge system with set of meanings M' , set of notations $S \cup \{\text{something}\}$, where *something* is a new denotation that denotes every element of the set $M \setminus M$, derivation rules of K , and notation-denotation relation of K extended by new pairs $(\text{something}, x)$ for every $x \in M \setminus M$.

Remark. A conservative extension of a knowledge system $K = \langle Q, D \rangle$ defined as a pair of a semantic system Q and a set of derivation rules D is $\langle Q', D \rangle$, where Q' is a primitive extension of Q .

Theorem 5. An object z is derivable from objects x, \dots, y in a conservative extension K' of a knowledge system K iff it is derivable from the same objects in K .

Proof: The derivation rules of K are neither applicable to *something*, nor can they produce *something*, hence the derivability in K' is the same as in K .

This theorem says that notation *something* can denote many things -- it introduces an open world. However, one can reason only about the known part of the world, and does not reason about the meanings that are indistinguishable in the knowledge system.

5 Operations on knowledge systems

Like in formal languages, one can define a variety of operations (e.g. sum and product) between semantically connected knowledge systems. Sum and product of knowledge systems are defined as follows.

Definition 14. *Sum of knowledge systems* K_1, K_2 denoted by \oplus is a knowledge system over the set of meanings $M_1 \cup M_2$ with the set of notations $S_1 \cup S_2$, the notation-denotation relation $f_1 \cup f_2$ and the set of derivation rules $D_1 \cup D_2$, where the sets $S_1, S_2, f_1, f_2, D_1, D_2$ are obtained from $S_1, S_2, f_1, f_2, D_1, D_2$ by renaming the notations with contradictory meanings in $S_1 \cap S_2$ so that the contradiction disappears.

Definition 15. *Product of knowledge systems* K_1 and K_2 denoted by \otimes is a knowledge system with the set of notations $S_1 \cap S_2$, set of meanings $M_1 \cap M_2$, the set of derivation rules $D_1 \cap D_2$, the notation-denotation relation $f_1 \cap f_2$, where D_1, D_2 are obtained from D_1, D_2 by excluding all rules that include notations not belonging to $S_1 \cap S_2$.

6 Lattice of goals

It is assumed that a knowledge system can be used for answering questions or, more generally, finding solutions for given goals. Let us have a finite set G of atomic goals defined on a knowledge system K . We do not restrict a meaning of an atomic goal. The only restriction is that an atomic goal has to be defined only in terms of notations, i.e. on the syntactic level. In particular, it can be just a notation that has to be derived in the knowledge system. An atomic goal can be solved/satisfied on a knowledge system by using the existing knowledge and inference engine of the system. Usually not all atomic goals can be solved, and a set of solvable atomic goals for a knowledge system can be defined.

Definition 16. A goal on a knowledge system K is a subset of the set G of atomic goals of the knowledge system K . A goal is solved iff all its elements (atomic goals) are solved.

Theorem 6. The set of goals constitutes a Boolean lattice with the set theoretic operations union \cup and intersection \cap .

Proof. A set of subsets of a finite set constitutes a Boolean lattice.

The definition of an atomic goal is quite abstract. We give here an example of a set of atomic goals in order to explain the meaning of this definition. Let us have a knowledge system of computability of values of variables, where the set of variables is a finite set V . The set of atomic goals will be $G = \{ w \rightarrow v \mid w \subseteq V \ \& \ v \in V \}$. An atomic goal $w \rightarrow v$ has the meaning "compute a value of v from the values of elements of w ".

7 Lattice of knowledge systems

The following shows that consistent semantic systems (pre-ontologies) play an important role in building knowledge systems. They enable one to avoid the contradictory meanings of elements when a union $S_1 \cup S_2$ of notations of knowledge systems is built, and make the results of the sum of knowledge systems easily observable. A sum of knowledge systems with consistent semantic systems will always have a set of notations $S_1 \cup S_2$, a notation-denotation relation $f_1 \cup f_2$ and a set of derivation rules $D_1 \cup D_2$, because operating in a consistent semantic system (pre-ontology) guarantees that $S_1 \cup S_2$ will not have elements with contradictory meanings. This gives us a nice symmetry of operations \oplus and \otimes .

Naturally, we could agree that we work in one and the same ontology, because this will also guarantee the consistency of notations. However, this would be an unnecessary strong restriction. The following presents this in exact terms.

Definition 17. Let us have a consistent semantic system O with a set of notations S , a set of meanings M and a notation-denotation relation f . Knowledge systems K_1, K_2 with the sets of notations $S_1 \subseteq S, S_2 \subseteq S$, sets of meanings $f(S_1), f(S_2)$, whose notation-denotation relations are restrictions of f on S_1 and S_2 are called *ontologically consistent* with respect to the semantic system O .

Theorem 7. Sum of ontologically consistent knowledge systems K_1 and K_2 has the set of notations $S_1 \cup S_2$, the notation-denotation relation $f_1 \cup f_2$ and the set of derivation rules $D_1 \cup D_2$.

Proof. The sets S_1 and S_2 do not have elements with different meanings, hence no renaming is performed.

Theorem 8. A set of ontologically consistent knowledge systems closed for operations of sum and products is a lattice with a zero element and a relation of partial order \leq defined as $K_1 \leq K_2$ iff $S_1 \subseteq S_2$ and $f_1 \subseteq f_2$.

Proof. It follows from the definitions of sum and product, and the fact that, due to the ontological consistency of knowledge systems, no renaming is needed.

An example could be a lattice of number systems. A pre-ontology is the set of numbers (both -- notations and meanings). Knowledge systems are the systems of natural numbers, positive integers, integers, positive rational numbers, rational numbers, positive real numbers and real numbers. It is quite obvious how these systems constitute a semilattice.

Theorem 9. If a goal g is solvable on a knowledge system K , and there is a knowledge system K' such that $K \leq K'$, then it is solvable on K' .

Proof. First we prove that g is also a goal on K' . Indeed, K' has all notations of K , and therefore g is defined on K' as well. The goal g is solvable on K' , because all derivation rules of K are available in K' as well.

Corollary. For every solvable goal there exist minimal knowledge systems that solve it.

Theorem 10. If a goal g is solvable on a knowledge system K , then any goal $g' \leq g$ is solvable on K .

Proof. A goal is solvable, iff all its elements (atomic goals) are solvable. g' includes only atomic goals that belong also to g , and therefore are solvable.

Corollary. For every knowledge system there are maximal solvable goals on it. Any solvable goal can be solved by some maximal solvable goal.

The Theorems 9 and 10 establish a nice correspondence between the lattice of goals and the lattice of

knowledge systems. These theorems demonstrate also the importance of (pre-)ontologies in the knowledge handling, because they are applicable only for ontologically consistent knowledge systems.

The Theorems 9 and 10 can explain the essence of the well known divide-and-conquer method of problem solving. If one has an unsolvable problem, i.e. an unsolvable goal, it must be divided into smaller ones solved in one and the same pre-ontology, but perhaps, on different knowledge systems. This may solve the original problem, if the pre-ontology is selected properly. Let us look at a known example of a ballistic pendulum. A bullet with known speed and mass hits a wooden block hanging as a pendulum, and fastens in it. We can measure an effect of the displacement of the pendulum with the bullet, using a geometric knowledge system, and find the change of its potential energy by using some knowledge system of physics. Using a knowledge system of mechanics, we can calculate the kinetic energy of the bullet. At the first glance, we could expect that these energies are equal, but they are not. We need one more knowledge system (thermal physics) that explains the loss of energy -- we can measure the amount of created heat, and find that now the balance of energies is restored.

8 Concluding remarks

We have defined knowledge as a notation with meaning, and have built on this principally simple definition the set of concepts for describing knowledge systems, knowledge handling and using knowledge for reaching goals. First, knowledge representation has been described by a semantic system -- an algebraic system with notation-denotation relation. The semantic systems are closely related to ontologies. They are pre-ontologies, i.e. ontologies without inference possibilities. An essential property of semantic systems is the absence of contradictory meanings of notations. To guarantee this property, an operation called harmonization has been introduced. A lattice of semantic systems has been described. A central concept of the work -- the concept of knowledge system has been defined in two different ways. First, following the ideas of S. Maslov, it has been defined as a free deductive system with interpretation. Second, it has been defined as a semantic system supplied with derivation rules. Both definitions give principally the same result. Ontologically consistent (i.e. harmonized) knowledge systems can also constitute lattices. These lattices are related to lattices of goals on the knowledge systems through solvability of goals.

Acknowledgements. We thank the Cooperative Cyber Defence Center of Excellence for the support to the present work.

9 References

- [1] M. Firebaugh. *Artificial Intelligence, A Knowledge-Based Approach*, Boyd&Fraser, 1988.
- [2] F. Gobet, *Is Experts' Knowledge Modular?* in: *Proceedings of the 23rd Meeting of the Cognitive Science Society* (336-431), Edinburgh, Scotland, 2000.
- [3] *Hierarchical Knowledge System*, US Patent #: 4,59, 1983.
- [4] P. Lorents, *Formalization of data and knowledge based on the fundamental notation-denotation relation*, in *IC-AI'2001 International Conference. Proceedings. vol. III. (1297 – 1301)* CSREA Press, Las Vegas, 2002.
- [5] P. Lorents, D. Lorents, *Intelligence and the Notation-Denotation Relation*, in: *IC-AI'2003 vol. III. (703 – 707)*, CSREA Press, Las Vegas, 2003.
- [6] S. Maslov. *Theory of Deductive Systems and its Application*. The MIT Press, 1987.
- [7] E. Tyugu. *Understanding Knowledge Architectures. Knowledge-Based Systems*. Vol. 19, No. 1, (50–56), 2006 .
- [8] A. Newell, R Young, T. Poek. *The approach through symbols, in the simulation of Human Intelligence* (D. Broadbent, ed), Blackwell Publish., 1993.

**PROCEEDINGS OF
THE 2009 INTERNATIONAL CONFERENCE ON
ARTIFICIAL INTELLIGENCE**

ICAI 2009

Volume II

Editors

**Hamid R. Arabnia
David de la Fuente, Jose A. Olivas**

Associate Editors

**Elena B. Kozerenko, Peter M. LaMonica
Raymond A. Liuzzi, Ashu M. G. Solo**



WORLD COMP'09

July 13-16, 2009

Las Vegas Nevada, USA

www.world-academy-of-science.org

©CSREA Press

This set of volumes contain papers presented at The 2009 International Conference on Artificial Intelligence (ICAI'09). Their inclusion in this publication does not necessarily constitute endorsements by editors or by the publisher.

Copyright and Reprint Permission

Copying without a fee is permitted provided that the copies are not made or distributed for direct commercial advantage, and credit to source is given. Abstracting is permitted with credit to the source. Please contact the publisher for other copying, reprint, or republication permission.

Copyright © 2009 CSREA Press
ISBN: 1-60132-107-4, 1-60132-108-2 (1-60132-109-0)
Printed in the United States of America

CSREA Press
U. S. A.

**PROCEEDINGS OF
THE 2009 INTERNATIONAL CONFERENCE ON
ARTIFICIAL INTELLIGENCE**

ICAI 2009

Volume II

Editors

**Hamid R. Arabnia
David de la Fuente, Jose A. Olivas**

Associate Editors

**Elena B. Kozerenko, Peter M. LaMonica
Raymond A. Liuzzi, Ashu M. G. Solo**



WORLDCOMP'09

July 13-16, 2009

Las Vegas Nevada, USA

www.world-academy-of-science.org

© CSREA Press